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LETTER TO THE EDITOR

Application of directed lattice animal theory to river networks

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Abstract. It is shown that the theory of directed lattice animals is a good candidate for determining the exponents which describe river networks.

In this Letter we investigate the possible connection between the relation of stream length to drainage area of river networks (see for instance Hack 1957) and the theory of directed lattice animals. We begin with a brief description of directed lattice animals and then describe how this can be related to the properties of river networks.

There have been several recent studies of directed lattice animals (Day and Lubensky 1982, Dhar *et al* 1982, Lubensky and Vannimenus 1982, Nadal *et al* 1982, Redner and Coniglio 1982, Redner and Yang 1982) and there are good results for critical exponents particularly in two dimensions. We are interested in single root bond lattice animals, i.e. finite clusters of directed bonds on a lattice which grow from a single point. In figure 1 we show an animal of this type on a square lattice. The bonds are allowed only to point either to the right or up and it can easily be seen that the animal grows preferentially along the (1, 1) direction. If we define the length of the cluster along the (1, 1) direction to be ξ_{\parallel} and the width to be ξ_{\perp} then as the total number of bonds N in the animal becomes large

$$\xi_{\parallel} \sim N^{\nu_{\parallel}} \quad \xi_{\perp} \sim N^{\nu_{\perp}}, \quad (1)$$

for a *typical* animal. Another interesting length in the problem is that of the longest directed path through the bonds L (see figure 1). For a square lattice we can easily see that L is given by

$$L = \sqrt{2}\xi_{\parallel} \quad (2)$$

which implies that

$$L \sim N^{\nu_{\parallel}}. \quad (3)$$

How may directed lattice animal calculations apply to river networks? If we look again at figure 1 we see that for a large directed lattice animal, its shape would seem similar to that of a river network. In this case we identify the root of the lattice animal with the 'mouth' of the river and the preferred direction with the slope of the land. Figure 1 contains more 'islands' than is likely to be typical of river networks. However, directed lattice animals without any closed loops are in the same universality class as those of figure 1 (Nadal *et al* 1982).

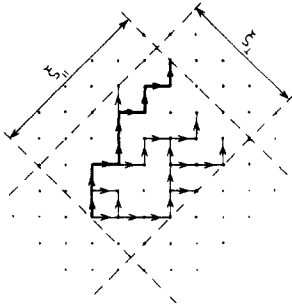


Figure 1. The diagram shows a (1, 1) directed lattice animal on the square lattice. The bold line shows the longest path and the broken lines show how the lengths ξ_{\parallel} and ξ_{\perp} are determined. The arrows point upstream from the mouth.

Hack (1957) has made an extensive study of the relation between the length L of the longest river of a drainage region of area A . The term length denotes the distance from a locality on a stream to the drainage divide at the head of the longest stream above it. The measurement was made generally on maps or aerial photographs with a map measure, along the stream channel and following meanders and bends; but in a few drainage basins it was made by tape traverse. The area A refers to the drainage area above the particular locality, including the drainage basin of the principal stream and of all the tributaries which enter it above the locality. In practice, area was measured on topographic maps or in a few cases from aerial photographs, by means of a planimeter. Regardless of the geologic conditions, it was found that at least in the northeastern United States

$$L \sim cA^{D/2} \quad (4)$$

where L is the length in miles, A is the area in square miles and the exponent $D = 1.2$. (However, measurements of streams eroding bedrock areas in the Mingus Mountain quadrangle, Arizona, indicated that the exponent D might not be completely universal as there it was found to be 1.4.) The coefficient c in equation (4) averages at 1.4 but ranges between 1 and 2.5. It is certainly not universal as, for example, in sandstone areas it averages around 2.

If the analogy between river networks and directed lattice animals is valid, then one can calculate D as follows.

$$\text{Area of basin } A \sim \xi_{\parallel}\xi_{\perp} \sim N^{\nu_{\parallel} + \nu_{\perp}}, \quad (5)$$

$$\text{Length of longest stream } L \sim N^{\nu_{\parallel}}$$

which implies

$$D = 2\nu_{\parallel}/(\nu_{\parallel} + \nu_{\perp}). \quad (6)$$

Notice that for any model in which $\nu_{\parallel} = \nu_{\perp}$, D would take its 'natural' value of unity. We have calculated D from a variety of the available sets of values for ν_{\parallel} and ν_{\perp} for two-dimensional lattice animals and the results are shown in table 1. The value of ν_{\perp} in two dimensions is believed to be exactly $\frac{1}{2}$ (Lubensky and Vannimenus 1982, Nadal *et al* 1982) and, therefore, the second three results should be more reliable. All these lie within the 1.2–1.4 range for river networks reported by Hack. Hack's data was obtained for basins whose areas A ranged from 0.12 to 375 square miles.

Table 1. Values of D for two-dimensional lattice animals.

	ν_{\perp}	ν_{\parallel}	$D = \frac{2\nu_{\parallel}}{\nu_{\parallel} + \nu_{\perp}}$
Mean field theory	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{4}{3}$
Flory method ^{a,b}	$\frac{9}{16}$	$\frac{13}{16}$	1.18
ε -expansion ^c	$\frac{1}{4} + \frac{1}{36}\varepsilon$	$\frac{1}{2} + \frac{1}{24}\varepsilon$	$\frac{4}{3}(1 - \frac{1}{108}\varepsilon) = 1.27$ for $\varepsilon = 5$
Adapted Flory method ^b	$\frac{1}{2}$	$\frac{13}{16}$	1.24
Series data ^d	0.5	0.800	1.23
Transfer matrix and phenomenological renormalisation ^e	$\frac{1}{2}$	$\frac{9}{11}$	1.28
Measurement of real river networks ^f	—	—	1.2–1.4

^aRedner and Coniglio 1982

^bLubensky and Vannimenus 1982

^cDay and Lubensky 1982

^dRedner and Yang 1982

^eNadal *et al* 1982

^fHack 1957.

Mandelbrot (1977) reports that for very large basins (area $>10^4$ km²) the Hack relation fails and D for such basins goes down to 1. Large basins tend to have smaller slopes so that 'directed' animals are less relevant and a crossover should take place to a non-directed (and, therefore, isotropic) animal with $\nu_{\parallel} = \nu_{\perp}$ and $D = 1$.

It thus seems likely that (directed) lattice animals are a good candidate for describing a model of river networks.

We should like to acknowledge useful discussions with Alan Bray.

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